Acta Arith. 159(2013), no.2, 169-200

Legendre polynomials and supercongruences

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Abstract

Let $p>3$ be a prime, and let R_p be the set of rational numbers whose denominator is not divisible by p. Let $\{P_n(x)\}$ be the Legendre polynomials. In this paper we mainly show that for $m,n,t\in R_p$ with $m\not\equiv 0\pmod p$,

$$P_{[\frac p6]}(t)\equiv -\Big(\frac 3p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3x+2t}{p}\Big)\pmod p$$

and

$$\Big(\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\Big)^2 \equiv \Big(\frac{-3m}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{4m^3+27n^2}{12^3\cdot 4m^3}\Big)^k\pmod p,$$

where $(\frac ap)$ is the Legendre symbol and $[x]$ is the greatest integer function. As an application we solve some conjectures of Z.W. Sun and the author concerning $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}/m^k \pmod{p^2}$, where m is an integer not divisible by p.

MSC: Primary 11A07, Secondary 33C45, 11E25, 11L10, 05A10, 05A19
Keywords: Binomial coefficient; congruence; Legendre polynomial; character sum; binary quadratic form

1. Introduction

Let $\{P_n(x)\}$ be the Legendre polynomials given by $P_0(x)=1$, $P_1(x)=x$ and $(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$ $(n\ge 1)$. It is well known that (see [B, p. 151], [G, (3.132)-(3.133)])

(1.1)

$$P_n(x)=\frac{1}{2^n}\sum_{k=0}^{[n/2]}\binom nk(-1)^k\binom{2n-2k}{n}x^{n-2k}=\frac{1}{2^n\cdot n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n,$$

where $[a]$ is the greatest integer not exceeding a. From (1.1) we see that

(1.2)

$$P_n(-x)=(-1)^nP_n(x),\ P_{2m+1}(0)=0 \text{ and } P_{2m}(0)=\frac{1}{(-4)^m}\binom{2m}{m}.$$

We also have the following formula due to Murphy ([G, (3.135)]):

(1.3)

$$P_n(x)=\sum_{k=0}^n\binom nk\binom{n+k}k\Big(\frac{x-1}2\Big)^k=\sum_{k=0}^n\binom{2k}k\binom{n+k}{2k}\Big(\frac{x-1}2\Big)^k.$$

We remark that $\binom nk\binom{n+k}k=\binom{2k}k\binom{n+k}{2k}$.

Let $\mathbb Z$ be the set of integers, and for a prime p let R_p be the set of rational numbers whose denominator is not divisible by p. Let $(\frac am)$ be the Jacobi symbol. In [S4-S6] the author showed that for any prime $p>3$ and $t\in R_p$,

$$(1.4)\ P_{\frac{p-1}2}(t)\equiv-\Big(\frac{-6}p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3(t^2+3)x+2t(t^2-9)}p\Big)\pmod p,$$

$$(1.5)\ P_{[\frac p3]}(t)\equiv-\Big(\frac p3\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3+3(4t-5)x+2(2t^2-14t+11)}p\Big)\pmod p,$$

$$(1.6)\ P_{[\frac p4]}(t)\equiv-\Big(\frac 6p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-\frac{3(3t+5)}2x+9t+7}p\Big)\pmod p.$$

In this paper, by using elementary arguments, we prove that for any prime $p>3$ and $t\in R_p$,

$$(1.7)\qquad P_{[\frac p6]}(t)\equiv-\Big(\frac 3p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3x+2t}p\Big)\pmod p.$$

Moreover, for $m,n\in R_p$ with $m\not\equiv0\pmod p$ we have

$$(1.8)\quad \sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}p\Big)\equiv\begin{cases}-(-3m)^{\frac{p-1}4}P_{[\frac p6]}\big(\frac{3n\sqrt{-3m}}{2m^2}\big)\pmod p&\text{if } p\equiv1\pmod 4,\\ -\frac{(-3m)^{\frac{p+1}4}}{\sqrt{-3m}}P_{[\frac p6]}\big(\frac{3n\sqrt{-3m}}{2m^2}\big)\pmod p&\text{if } p\equiv3\pmod 4\end{cases}$$

and

$$\Big(\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}p\Big)\Big)^2\equiv\Big(\frac{-3m}p\Big)\sum_{k=0}^{[p/6]}\binom{2k}k\binom{3k}k\binom{6k}{3k}\Big(\frac{4m^3+27n^2}{12^3\cdot4m^3}\Big)^k\pmod p.$$

It is well known (see for example [S2, pp.221-222]) that the number of points on the curve $y^2 = x^3 + mx + n$ over the field $\mathbb{F}_p$ with p elements is given by

$$\#E_p(x^3 + mx + n) = p + 1 + \sum_{x=0}^{p-1} \Big(\frac{x^3 + mx + n}{p}\Big).$$

For positive integers a, b and n, if $n = ax^2 + by^2$ for some integers x and y, we briefly say that $n = ax^2 + by^2$. Recently the author's brother Zhi-Wei Sun[Su1,Su3] and the author[S4] posed some conjectures for $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}/m^k$ modulo p^2, where $p > 3$ is a prime and $m \in \mathbb{Z}$ with $p \nmid m$. For example, Zhi-Wei Sun ([Su3, Conjecture 2.8]) conjectured that for any prime $p > 3$,

(1.9)

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}} \equiv \begin{cases} 0 \pmod{p^2} & \text{if } (\frac{p}{19}) = -1, \\ (\frac{-6}{p})(x^2 - 2p) \pmod{p^2} & \text{if } (\frac{p}{19}) = 1 \text{ and so } 4p = x^2 + 19y^2. \end{cases}$$

Using (1.8) and known character sums we determine $P_{[\frac{p}{6}]}(x) \pmod p$ for 11 values of x (see Corollaries 2.1-2.7), and $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}/m^k \pmod{p^2}$ for $m = -15^3, 20^3, -32^3, 2\cdot 30^3, 66^3, -96^3, -3\cdot 160^3, 255^3, -960^3, -5280^3, -640320^3$. Thus we solve some conjectures in [Su1, Su3] and [S4]. For example, we confirm (1.9) in the case $(\frac{p}{19}) = -1$ and prove it when $(\frac{p}{19}) = 1$ and the modulus is p.

Let p be a prime greater than 3. In the paper we also determine $\sum_{k=0}^{p-1}\binom{3k}{k}\binom{6k}{3k}/864^k \pmod{p^2}$ and establish the general congruence

(1.10)

$$\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}(x(1-432x))^k \equiv \Big(\sum_{k=0}^{p-1}\binom{3k}{k}\binom{6k}{3k}x^k\Big)^2 \pmod{p^2}.$$

2. Congruences for $P_{[\frac{p}{6}]}(t) \pmod p$

Lemma 2.1. *Let p be an odd prime. Then*

(i) $\binom{\frac{p-1}{2}}{k} \equiv \frac{1}{(-4)^k}\binom{2k}{k} \pmod p$ *for* $k = 0, 1, \dots, \frac{p-1}{2}$,

(ii) $\binom{\frac{p-1}{2} - k}{2k} \equiv \frac{\binom{6k}{3k}\binom{3k}{k}}{4^{2k}\binom{2k}{k}} \pmod p$ *for* $k = 0, 1, \dots, [\frac{p}{6}]$,

(iii) $\binom{[\frac{p}{3}] + k}{2k} \equiv \frac{1}{(-27)^k}\binom{3k}{k} \pmod p$ *for $p \ne 3$ and* $k = 0, 1, \dots, [\frac{p}{3}]$.

Proof. For $k \in \{0, 1, \dots, \frac{p-1}{2}\}$ we have $\binom{\frac{p-1}{2}}{k} \equiv \binom{-\frac{1}{2}}{k} = \frac{1}{(-4)^k}\binom{2k}{k} \pmod p$. Thus (i) holds. Now suppose $k \in \{0, 1, \dots, [\frac{p}{6}]\}$. It is clear that

$$\binom{\frac{p-1}{2}-k}{2k} = \frac{\frac{p-1-2k}{2}\cdot\frac{p-3-2k}{2}\cdots\frac{p-(6k-1)}{2}}{(2k)!} \equiv \frac{(2k+1)(2k+3)\cdots(6k-1)}{(-2)^{2k}\cdot(2k)!}$$
$$= \frac{(6k)!}{4^k(2k)!^2(2k+2)(2k+4)\cdots(6k)} = \frac{(6k)!}{4^k(2k)!^2\cdot 2^{2k}\cdot\frac{(3k)!}{k!}}$$
$$= \frac{\binom{6k}{3k}\binom{3k}{k}}{4^{2k}\binom{2k}{k}} \pmod p.$$

Thus (ii) is true. (iii) was given by the author in [S4, proof of Lemma 2.3]. The proof is now complete.

Lemma 2.2. *Let $p>3$ be a prime and $k\in\{0,1,\ldots,[\frac{p}{12}]\}$. Then*

$$\binom{[\frac p6]}{k}\binom{2[\frac p6]-2k}{[\frac p6]}$$
$$\equiv (-1)^{[\frac p6]}3^{3k+\frac{1-(\frac p3)}{2}}4^{[\frac p6]-k}\binom{\frac{p-1}{2}}{[\frac p3]-k}\binom{\frac{p-(\frac p3)}{6}+k}{3k+\frac{1-(\frac p3)}{2}} \pmod p.$$

Proof. For $m,n,r\in\mathbb{Z}$ with $m\ge n\ge r\ge 0$ it is easily seen that $\binom mn\binom nr=\binom mr\binom{m-r}{n-r}$. Thus, using Lemma 2.1(i) we see that

$$\binom{[\frac p6]}{k}\binom{2[\frac p6]-2k}{[\frac p6]}$$
$$= \binom{[\frac p6]}{[\frac p6]-k}\binom{2[\frac p6]-2k}{[\frac p6]} = \binom{2([\frac p6]-k)}{[\frac p6]-k}\binom{[\frac p6]-k}{k}$$
$$\equiv (-4)^{[\frac p6]-k}\binom{\frac{p-1}{2}}{[\frac p6]-k}\binom{[\frac p6]-k}{k}$$
$$= (-4)^{[\frac p6]-k}\binom{\frac{p-1}{2}}{k}\binom{\frac{p-1}{2}-k}{[\frac p6]-2k}$$
$$\equiv (-4)^{[\frac p6]-2k}\binom{2k}{k}\frac{(\frac{p-1}{2}-k)!}{([\frac p6]-2k)!([\frac{p+1}{3}]+k)!} \pmod p.$$

If $p\equiv 1 \pmod 3$, using Lemma 2.1(i) we see that

$$\binom{\frac{p-1}{2}}{\frac{p-1}{3}-k}\binom{\frac{p-1}{6}+k}{3k} = \binom{\frac{p-1}{2}}{\frac{p-1}{6}+k}\binom{\frac{p-1}{6}+k}{3k} = \binom{\frac{p-1}{2}}{3k}\binom{\frac{p-1}{2}-3k}{\frac{p-1}{6}-2k}$$
$$\equiv \frac{1}{(-4)^{3k}}\binom{6k}{3k}\frac{(\frac{p-1}{2}-3k)!}{(\frac{p-1}{6}-2k)!(\frac{p-1}{3}-k)!} \pmod p.$$

Thus, from the above and Lemma 2.1 we deduce that

$$\frac{\binom{[\frac p6]}{k}\binom{2[\frac p6]-2k}{[\frac p6]}}{\binom{\frac{p-1}{2}}{\frac{p-1}{3}-k}\binom{\frac{p-1}{6}+k}{3k}}$$
$$\equiv (-4)^{[\frac p6]-2k+3k}\frac{\binom{2k}{k}(\frac{p-1}{2}-k)!(\frac{p-1}{3}-k)!}{\binom{6k}{3k}(\frac{p-1}{2}-3k)!(\frac{p-1}{3}+k)!}$$

$$= (-4)^{[\frac p6]+k}\frac{\binom{2k}{k}\binom{\frac{p-1}{2}-k}{2k}}{\binom{6k}{3k}\binom{\frac{p-1}{3}+k}{2k}} \equiv (-4)^{[\frac p6]+k}\frac{\binom{2k}{k}\binom{6k}{3k}\binom{3k}{k}/(4^{2k}\binom{2k}{k})}{\binom{6k}{3k}\binom{3k}{k}/(-27)^k}$$
$$= (-27)^k(-4)^{[\frac p6]-k} = (-1)^{[\frac p6]}3^{3k}4^{[\frac p6]-k} \pmod p.$$

If $p\equiv 2 \pmod 3$, using Lemma 2.1(i) we see that

$$\begin{aligned}&\binom{\frac{p-1}{2}}{\frac{p-2}{3}-k}\binom{\frac{p+1}{6}+k}{3k+1}\\ &= \binom{\frac{p-1}{2}}{\frac{p+1}{6}+k}\binom{\frac{p+1}{6}+k}{3k+1} = \binom{\frac{p-1}{2}}{3k+1}\binom{\frac{p-3}{2}-3k}{\frac{p-5}{6}-2k}\\ &\equiv \frac{1}{(-4)^{3k+1}}\binom{6k+2}{3k+1}\frac{(\frac{p-3}{2}-3k)!}{(\frac{p-5}{6}-2k)!(\frac{p-2}{3}-k)!}\\ &\equiv \frac{1}{(-4)^{3k}(3k+1)}\binom{6k}{3k}\frac{(\frac{p-1}{2}-3k)!}{(\frac{p-5}{6}-2k)!(\frac{p-2}{3}-k)!} \pmod p.\end{aligned}$$

Thus, from the above and Lemma 2.1 we deduce that

$$\begin{aligned}&\frac{\binom{[\frac p6]}{k}\binom{2[\frac p6]-2k}{[\frac p6]}}{\binom{\frac{p-1}{2}}{\frac{p-2}{3}-k}\binom{\frac{p+1}{6}+k}{3k+1}}\\ &\equiv 3(-4)^{[\frac p6]-2k+3k}\frac{\binom{2k}{k}(\frac{p-1}{2}-k)!(\frac{p-2}{3}-k)!}{\binom{6k}{3k}(\frac{p-1}{2}-3k)!(\frac{p-2}{3}+k)!}\\ &= 3(-4)^{[\frac p6]+k}\frac{\binom{2k}{k}\binom{\frac{p-1}{2}-k}{2k}}{\binom{6k}{3k}\binom{\frac{p-2}{3}+k}{2k}} \equiv 3(-4)^{[\frac p6]+k}\frac{\binom{2k}{k}\binom{6k}{3k}\binom{3k}{k}/(4^{2k}\binom{2k}{k})}{\binom{6k}{3k}\binom{3k}{k}/(-27)^k}\\ &= 3(-27)^k(-4)^{[\frac p6]-k} = (-1)^{[\frac p6]}3^{3k+1}4^{[\frac p6]-k} \pmod p.\end{aligned}$$

This completes the proof.

Theorem 2.1. *Let $p>3$ be a prime and $m,n\in R_p$ with $m\not\equiv 0 \pmod p$. Then*

$$\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big) \equiv \begin{cases} -(-3m)^{\frac{p-1}{4}}P_{[\frac p6]}\big(\frac{3n\sqrt{-3m}}{2m^2}\big) \pmod p & \text{if } p\equiv 1 \pmod 4,\\ -\frac{(-3m)^{\frac{p+1}{4}}}{\sqrt{-3m}}P_{[\frac p6]}\big(\frac{3n\sqrt{-3m}}{2m^2}\big) \pmod p & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Proof. For any positive integer k it is well known (see [IR, Lemma 2, p.235]) that

$$\sum_{x=0}^{p-1}x^k \equiv \begin{cases} p-1 \pmod p & \text{if } p-1\mid k,\\ 0 \pmod p & \text{if } p-1\nmid k.\end{cases}$$

For $k,r\in\mathbb{Z}$ with $0\le r\le k\le \frac{p-1}{2}$ we have $0\le k+2r\le \frac{3(p-1)}{2}$. Thus,

$$\sum_{x=0}^{p-1}x^{k+2r}\equiv\begin{cases}p-1\pmod p & \text{if } k=p-1-2r,\\ 0\pmod p & \text{if } k\ne p-1-2r\end{cases}$$

and therefore

(2.1)

$$\begin{aligned}&\sum_{x=0}^{p-1}(x^3+mx+n)^{\frac{p-1}{2}}\\
&=\sum_{x=0}^{p-1}\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}(x^3+mx)^kn^{\frac{p-1}{2}-k}\\
&=\sum_{x=0}^{p-1}\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}\sum_{r=0}^{k}\binom kr x^{3r}(mx)^{k-r}n^{\frac{p-1}{2}-k}\\
&=\sum_{r=0}^{(p-1)/2}\sum_{k=r}^{(p-1)/2}\binom{(p-1)/2}{k}\binom kr m^{k-r}n^{\frac{p-1}{2}-k}\sum_{x=0}^{p-1}x^{k+2r}\\
&\equiv(p-1)\sum_{r=0}^{(p-1)/2}\binom{(p-1)/2}{p-1-2r}\binom{p-1-2r}{r}m^{p-1-3r}n^{2r-\frac{p-1}{2}}\\
&=(p-1)\sum_{\frac{p-1}{4}\le r\le\frac{p-1}{3}}\binom{(p-1)/2}{p-1-2r}\binom{p-1-2r}{r}m^{p-1-3r}n^{2r-\frac{p-1}{2}}\pmod p.\end{aligned}$$

If $n\equiv 0\pmod p$, from the above we deduce that

$$\begin{aligned}\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)&\equiv\sum_{x=0}^{p-1}(x^3+mx+n)^{\frac{p-1}{2}}\\
&\equiv\begin{cases}-\binom{\frac{p-1}{2}}{\frac{p-1}{4}}m^{\frac{p-1}{4}}\pmod p & \text{if } 4\mid p-1,\\ 0\pmod p & \text{if } 4\mid p-3.\end{cases}\end{aligned}$$

Thus applying (1.2) and Lemma 2.2 (with $k=[\frac{p}{12}]$) we get

$$P_{[\frac p6]}(0)=\begin{cases}\frac{1}{(-4)^{[\frac{p}{12}]}}\binom{[\frac p6]}{[\frac{p}{12}]}\equiv(-1)^{[\frac{p}{12}]}3^{\frac{p-1}{4}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}\\ \qquad\equiv(-3)^{-\frac{p-1}{4}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}}\pmod p & \text{if } 4\mid p-1,\\ 0 & \text{if } 4\mid p-3.\end{cases}$$

Hence the result is true for $n\equiv 0\pmod p$.

Now we assume $n\not\equiv 0\pmod p$. From (2.1) we see that

$$\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)$$

$$\equiv \sum_{x=0}^{p-1}(x^3+mx+n)^{\frac{p-1}{2}}$$

$$\equiv (p-1)\frac{m^{p-1}}{n^{\frac{p-1}{2}}}\sum_{\frac{p-1}{4}\le r\le \frac{p-1}{3}}\binom{\frac{p-1}{2}}{p-1-2r}\binom{p-1-2r}{r}\frac{n^{2r}}{m^{3r}}$$

$$\equiv -\Big(\frac{n}{p}\Big)\sum_{\frac{p-1}{4}\le r\le \frac{p-1}{3}}\binom{(p-1)/2}{r}\binom{\frac{p-1}{2}-r}{p-1-3r}\Big(\frac{n^2}{m^3}\Big)^r$$

$$= -\Big(\frac{n}{p}\Big)\sum_{k=0}^{[\frac{p}{12}]}\binom{\frac{p-1}{2}}{[\frac{p}{3}]-k}\binom{\frac{p-(\frac{p}{3})}{6}+k}{3k+\frac{1-(\frac{p}{3})}{2}}\Big(\frac{n^2}{m^3}\Big)^{[\frac{p}{3}]-k} \pmod p.$$

On the other hand, by (1.1),

$$P_{[\frac{p}{6}]}(\frac{3n\sqrt{-3m}}{2m^2})$$

$$= 2^{-[\frac{p}{6}]}\sum_{k=0}^{[\frac{p}{12}]}\binom{[\frac{p}{6}]}{k}(-1)^k\binom{2[\frac{p}{6}]-2k}{[\frac{p}{6}]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big)^{[\frac{p}{6}]-2k}$$

$$= 2^{-[\frac{p}{6}]}\sum_{k=0}^{[\frac{p}{12}]}\binom{[\frac{p}{6}]}{k}(-1)^k\binom{2[\frac{p}{6}]-2k}{[\frac{p}{6}]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big)^{\frac{1-(\frac{-1}{p})}{2}}\Big(-\frac{27n^2}{4m^3}\Big)^{[\frac{p}{12}]-k}$$

$$= (-1)^{[\frac{p}{12}]}2^{-[\frac{p}{6}]}$$

$$\times\sum_{k=0}^{[\frac{p}{12}]}\binom{[\frac{p}{6}]}{k}\binom{2[\frac{p}{6}]-2k}{[\frac{p}{6}]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big)^{\frac{1-(\frac{-1}{p})}{2}}\Big(\frac{27n^2}{4m^3}\Big)^{[\frac{p}{3}]-k-\frac{p-(\frac{-1}{p})}{4}}$$

$$\equiv \delta(m,p)^{-1}\Big(\frac{n}{p}\Big)\Big(\frac{3}{p}\Big)(-1)^{[\frac{p}{12}]}2^{-[\frac{p}{6}]}$$

$$\times\sum_{k=0}^{[\frac{p}{12}]}\binom{[\frac{p}{6}]}{k}\binom{2[\frac{p}{6}]-2k}{[\frac{p}{6}]}\Big(\frac{27n^2}{4m^3}\Big)^{[\frac{p}{3}]-k} \pmod p,$$

where

$$\delta(m,p)=\begin{cases}(-3m)^{\frac{p-1}{4}} & \text{if } p\equiv 1 \pmod 4,\\ \dfrac{(-3m)^{\frac{p+1}{4}}}{\sqrt{-3m}} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Hence, by the above and Lemma 2.2 we get

$$\delta(m,p)P_{[\frac{p}{6}]}(\frac{3n\sqrt{-3m}}{2m^2})$$

$$\equiv\Big(\sum_{k=0}^{[\frac{p}{12}]}(-1)^{[\frac{p}{6}]}3^{3k+\frac{1-(\frac{p}{3})}{2}}4^{[\frac{p}{6}]-k}\binom{\frac{p-1}{2}}{[\frac{p}{3}]-k}\binom{\frac{p-(\frac{p}{3})}{6}+k}{3k+\frac{1-(\frac{p}{3})}{2}}\Big(\frac{27n^2}{4m^3}\Big)^{[\frac{p}{3}]-k}\Big)$$

$$\times\Big(\frac{n}{p}\Big)\Big(\frac{3}{p}\Big)(-1)^{[\frac{p}{12}]}2^{-[\frac{p}{6}]} \pmod p.$$

Since

$$\Big(\frac{3}{p}\Big)(-1)^{[\frac{p}{12}]}2^{-[\frac p6]}(-1)^{[\frac p6]}3^{3k+\frac{1-(\frac p3)}{2}}4^{[\frac p6]-k}\Big(\frac{27}{4}\Big)^{[\frac p3]-k}$$
$$=(-1)^{[\frac{p}{12}]+[\frac p6]}\Big(\frac 3p\Big)2^{[\frac p6]-2[\frac p3]}3^{3[\frac p3]+(1-(\frac p3))/2}$$
$$=(-1)^{[\frac{p}{12}]+[\frac p6]}\Big(\frac 3p\Big)2^{-\frac{p-1}{2}}3^{p-1}\equiv(-1)^{[\frac{p}{12}]+[\frac p6]}\cdot(-1)^{\frac{p-(\frac p3)}{6}}\cdot(-1)^{-[\frac{p+1}{4}]}$$
$$=(-1)^{2[\frac{p}{12}]}=1 \pmod p,$$

from the above we deduce that

$$\begin{aligned}&\delta(m,p)P_{[\frac p6]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big)\\&\equiv\Big(\frac np\Big)\sum_{k=0}^{[\frac{p}{12}]}\binom{\frac{p-1}{2}}{[\frac p3]-k}\binom{\frac{p-(\frac p3)}{6}+k}{3k+\frac{1-(\frac p3)}{2}}\Big(\frac{n^2}{m^3}\Big)^{[\frac p3]-k}\\&\equiv-\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big) \pmod p.\end{aligned}$$

This completes the proof.

Remark 2.1 The congruence (2.1) was given by the author in [S5].

Corollary 2.1. *Let $p\ne 2,3,11$ be a prime. Then*

$$P_{[\frac p6]}\Big(\frac{21\sqrt{33}}{121}\Big)\equiv\begin{cases}(\frac{33}{p})(-33)^{\frac{p-1}{4}}2a \pmod p\\ \qquad\qquad \textit{if } 4\mid p-1,\ p=a^2+b^2 \textit{ and } 4\mid a-1,\\ 0 \pmod p\quad \textit{if } p\equiv 3 \pmod 4.\end{cases}$$

Proof. By [S6, Corollary 2.1 (with $t=\frac79$) and (2.2)],

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3-11x+14}{p}\Big)=\Big(\frac 2p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-4x}{p}\Big)\\&=\begin{cases}(-1)^{\frac{p+3}{4}}2a & \text{if } 4\mid p-1,\ p=a^2+b^2 \text{ and } 4\mid a-1,\\ 0 & \text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned}\tag{2.2}$$

Thus, taking $m=-11$ and $n=14$ in Theorem 2.1 we obtain the result.

Corollary 2.2. *Let $p>5$ be a prime. Then*

$$P_{[\frac p6]}\Big(\frac{7\sqrt{10}}{25}\Big)\equiv\begin{cases}(-1)^{\frac d2}(\frac 5p)5^{\frac{p-1}{4}}2c \pmod p\\ \qquad\qquad \textit{if } 8\mid p-1,\ p=c^2+2d^2 \textit{ and } 4\mid c-1,\\ (\frac 5p)5^{\frac{p-3}{4}}2d\sqrt{10} \pmod p\\ \qquad\qquad \textit{if } 8\mid p-3,\ p=c^2+2d^2 \textit{ and } 4\mid d-1,\\ 0 \pmod p\quad \textit{if } p\equiv 5,7 \pmod 8.\end{cases}$$

Proof. Using [S5, Lemma 4.2] we see that

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3-30x+56}{p}\Big)\\&=\sum_{x=0}^{p-1}\Big(\frac{(-x)^3-30(-x)+56}{p}\Big)=\Big(\frac{-1}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-30x-56}{p}\Big)\\&=\begin{cases}(-1)^{\frac{p+7}{8}}(\frac{3}{p})2c & \text{if } p\equiv 1 \pmod 8,\ p=c^2+2d^2 \text{ and } 4\mid c-1,\\(-1)^{\frac{p-3}{8}}(\frac{3}{p})2c & \text{if } p\equiv 3 \pmod 8,\ p=c^2+2d^2 \text{ and } 4\mid c-1,\\0 & \text{if } p\equiv 5,7 \pmod 8.\end{cases}\end{aligned}$$

By [S3, p.1317],

$$2^{[\frac p4]}\equiv\begin{cases}(-1)^{\frac{c^2-1}{8}}=(-1)^{\frac{p-1}{8}+\frac d2} \pmod p\\ \qquad\text{if } p=c^2+2d^2\equiv 1 \pmod 8,\\(-1)^{\frac{c^2-1}{8}}\frac dc=(-1)^{\frac{p-3}{8}}\frac dc \pmod p\\ \qquad\text{if } p=c^2+2d^2\equiv 3 \pmod 8 \text{ with } 4\mid c-d.\end{cases}$$

Now taking $m=-30$ and $n=56$ in Theorem 2.1 and applying the above we deduce the result.

Corollary 2.3. *Let $p>5$ be a prime. Then*

$$P_{[\frac p6]}\Big(\frac{11\sqrt5}{25}\Big)\equiv\begin{cases}5^{\frac{p-1}{4}}(\frac 5p)2A \pmod p\\ \qquad \textit{if } 12\mid p-1,\ p=A^2+3B^2 \textit{ and } 3\mid A-1,\\-5^{\frac{p-3}{4}}(\frac 5p)2A\sqrt5 \pmod p\\ \qquad \textit{if } 12\mid p-7,\ p=A^2+3B^2 \textit{ and } 3\mid A-1,\\0 \pmod p \quad \textit{if } p\equiv 2 \pmod 3.\end{cases}$$

Proof. By [S2, Lemma 2.3] (or [S6, Corollary 2.1 (with $t=5/3$) and (2.3)]) we have

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-15x+22}{p}\Big)=\begin{cases}-2A & \text{if } 3\mid p-1,\ p=A^2+3B^2 \text{ and } 3\mid A-1,\\0 & \text{if } p\equiv 2 \pmod 3.\end{cases}\tag{2.3}$$

Thus, taking $m=-15$ and $n=22$ in Theorem 2.1 we obtain the result.

Corollary 2.4. *Let $p>5$ be a prime. Then*

$$P_{[\frac p6]}\Big(\frac{253\sqrt{10}}{800}\Big)$$

$$\equiv\begin{cases}-(\frac{10}{p})10^{\frac{p-1}{4}}L \pmod p \\ \qquad\qquad \textit{if } 12\mid p-1,\ 4p=L^2+27M^2 \textit{ and } 3\mid L-1, \\ (\frac{10}{p})10^{\frac{p-3}{4}}L\sqrt{10} \pmod p \\ \qquad\qquad \textit{if } 12\mid p-7,\ 4p=L^2+27M^2 \textit{ and } 3\mid L-1, \\ 0 \pmod p \quad \textit{if } p\equiv 2 \pmod 3.\end{cases}$$

Proof. From [S5, Corollary 3.3] we know that

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-120x+506}{p}\Big) = \begin{cases}(\frac{2}{p})L & \text{if } 3\mid p-1,\ 4p=L^2+27M^2 \text{ and } 3\mid L-1, \\ 0 & \text{if } p\equiv 2 \pmod 3.\end{cases} \tag{2.4}$$

Thus taking $m=-120$ and $n=506$ in Theorem 2.1 we deduce the result.

Corollary 2.5. *Let $p>7$ be a prime. Then*

$$P_{[\frac p6]}\Big(\frac{3\sqrt{105}}{25}\Big) \equiv\begin{cases}2(\frac{p}{15})15^{\frac{p-1}{4}}C \pmod p \\ \qquad\qquad \textit{if } p\equiv 1,9,25 \pmod{28},\ p=C^2+7D^2 \textit{ and } 4\mid C-1, \\ 2(\frac{p}{15})15^{\frac{p-3}{4}}D\sqrt{105} \pmod p \\ \qquad\qquad \textit{if } p\equiv 11,15,23 \pmod{28},\ p=C^2+7D^2 \textit{ and } 4\mid D-1, \\ 0 \pmod p \quad \textit{if } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Proof. Since $(-x-7)^3-35(-x-7)+98=-(x^3+21x^2+112x)$, from [R1,R2] we see that

(2.5)

$$\sum_{x=0}^{p-1}\Big(\frac{x^3-35x+98}{p}\Big)=(-1)^{\frac{p-1}{2}}\sum_{x=0}^{p-1}\Big(\frac{x^3+21x^2+112x}{p}\Big) = \begin{cases}(-1)^{\frac{p+1}{2}}2C(\frac{C}{7}) & \text{if } p\equiv 1,2,4 \pmod 7 \text{ and so } p=C^2+7D^2, \\ 0 & \text{if } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Suppose $p\equiv 1,2,4 \pmod 7$ and so $p=C^2+7D^2$. By [S3, p.1317],

(2.6)

$$7^{[\frac p4]}\equiv\begin{cases}(\frac{C}{7}) \pmod p & \text{if } p\equiv 1,9,25 \pmod{28} \text{ and } 4\mid C-1, \\ -(\frac{C}{7})\frac{D}{C} \pmod p & \text{if } p\equiv 11,15,23 \pmod{28} \text{ and } 4\mid D-1.\end{cases}$$

Now taking $m=-35$ and $n=98$ in Theorem 2.1 and applying all the above we deduce the result.

Corollary 2.6. *Let p be a prime such that $p\neq 2,3,5,7,17$.*

(i) *If* $p\equiv 3,5,6\pmod 7$, *then* $P_{[\frac p6]}(\frac{171\sqrt{1785}}{85^2})\equiv 0\pmod p$.

(ii) *If* $p\equiv 1,2,4\pmod 7$ *and so* $p=C^2+7D^2$ *for some* $C,D\in\mathbb{Z}$, *then*

$$P_{[\frac p6]}\Big(\frac{171\sqrt{1785}}{85^2}\Big)\equiv\begin{cases}(\frac{255}{p})255^{\frac{p-1}{4}}\cdot 2C\pmod p\\ \qquad\qquad\qquad\qquad \text{if } 4\mid p-1 \text{ and } 4\mid C-1,\\ -(\frac{255}{p})255^{\frac{p-3}{4}}\cdot 2D\sqrt{1785}\pmod p\\ \qquad\qquad\qquad\qquad \text{if } 4\mid p-3 \text{ and } 4\mid D-1.\end{cases}$$

Proof. From [W, p.296] we know that

$$\sum_{x=0}^{p-1}\Big(\frac{(x^2+6x+2)(3x^2+16x)}{p}\Big)$$
$$=\begin{cases}-2(\frac{-2}{p})(\frac C7)C-(\frac 3p) & \text{if } p\equiv 1,2,4\pmod 7 \text{ and } p=C^2+7D^2,\\ -(\frac 3p) & \text{if } p\equiv 3,5,6\pmod 7.\end{cases}$$

As $(x^2+6x+2)(3x^2+16x)=x^4(3+34/x+102/x^2+32/x^3)$, we see that

$$\sum_{x=0}^{p-1}\Big(\frac{(x^2+6x+2)(3x^2+16x)}{p}\Big)$$
$$=\sum_{x=1}^{p-1}\Big(\frac{3+34/x+102/x^2+32/x^3}{p}\Big)=\sum_{x=1}^{p-1}\Big(\frac{3+34x+102x^2+32x^3}{p}\Big)$$
$$=\Big(\frac 2p\Big)\sum_{x=1}^{p-1}\Big(\frac{6+17\cdot 4x+\frac{51}{4}(4x)^2+(4x)^3}{p}\Big)$$
$$=\Big(\frac 2p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3+\frac{51}{4}x^2+17x+6}{p}\Big)-\Big(\frac{12}{p}\Big)$$

and

$$\sum_{x=0}^{p-1}\Big(\frac{x^3+\frac{51}{4}x^2+17x+6}{p}\Big)$$
$$=\sum_{x=0}^{p-1}\Big(\frac{(x-\frac{17}{4})^3+\frac{51}{4}(x-\frac{17}{4})^2+17(x-\frac{17}{4})+6}{p}\Big)$$
$$=\sum_{x=0}^{p-1}\Big(\frac{x^3-\frac{595}{16}x+\frac{5586}{64}}{p}\Big)=\sum_{x=0}^{p-1}\Big(\frac{(\frac x4)^3-\frac{595}{16}\cdot\frac x4+\frac{5586}{64}}{p}\Big)$$
$$=\sum_{x=0}^{p-1}\Big(\frac{x^3-595x+5586}{p}\Big).$$

Now combining all the above we deduce that

$$\tag{2.7} \begin{aligned} &\sum_{x=0}^{p-1}\Big(\frac{x^3-595x+5586}{p}\Big)\\ &=\begin{cases} (-1)^{\frac{p+1}{2}}2C(\frac{C}{7}) & \text{if } p=C^2+7D^2\equiv 1,2,4 \pmod 7,\\ 0 & \text{if } p\equiv 3,5,6 \pmod 7.\end{cases}\end{aligned}$$

Taking $m=-595$ and $n=5586$ in Theorem 2.1 and then applying (2.7) and (2.6) we deduce the result.

Corollary 2.7. *Let $p\ne 2,3,11$ be a prime.*

(i) *If $p\equiv 2,6,7,8,10 \pmod{11}$, then $P_{[\frac p6]}(\frac{7}{32}\sqrt{22})\equiv 0 \pmod p$.*

(ii) *If $p\equiv 1,3,4,5,9 \pmod{11}$ and hence $4p=u^2+11v^2$ for some $u,v\in\mathbb{Z}$, then*

$$\begin{aligned} &P_{[\frac p6]}\Big(\frac{7}{32}\sqrt{22}\Big)\\ &\equiv\begin{cases} -2^{\frac{p-1}{4}}u \pmod p & \text{if } 4\mid p-1 \text{ and } 4\mid u-1,\\ (-2)^{\frac{p-1}{4}}u \pmod p & \text{if } 4\mid p-1 \text{ and } 8\mid u-2,\\ -2^{\frac{p-3}{4}}v\sqrt{22} \pmod p & \text{if } 4\mid p-3 \text{ and } 4\mid v-1,\\ (-2)^{\frac{p-3}{4}}v\sqrt{22} \pmod p & \text{if } 4\mid p-3 \text{ and } 8\mid v-2.\end{cases}\end{aligned}$$

Proof. It is known (see [RP] and [JM]) that

$$\tag{2.8} \begin{aligned} &\sum_{x=0}^{p-1}\Big(\frac{x^3-96\cdot 11x+112\cdot 11^2}{p}\Big)\\ &=\begin{cases} (\frac{3}{p})(\frac{u}{11})u & \text{if } (\frac{p}{11})=1 \text{ and so } 4p=u^2+11v^2,\\ 0 & \text{if } (\frac{p}{11})=-1.\end{cases}\end{aligned}$$

Thus applying Theorem 2.1 we deduce that

$$P_{[\frac p6]}(\frac{7}{32}\sqrt{22})\equiv\begin{cases} -(\frac{-2}{p})22^{\frac{p-1}{4}}(\frac{u}{11})u \pmod p\\ \qquad\qquad \text{if } (\frac{p}{11})=1,\ 4\mid p-1 \text{ and } 4p=u^2+11v^2,\\ -(\frac{-2}{p})22^{\frac{p-3}{4}}(\frac{u}{11})u\sqrt{22} \pmod p\\ \qquad\qquad \text{if } (\frac{p}{11})=1,\ 4\mid p-3 \text{ and } 4p=u^2+11v^2,\\ 0 \pmod p \text{ if } (\frac{p}{11})=-1.\end{cases}$$

Now assume $(\frac{p}{11})=1$ and so $4p=u^2+11v^2$. If $u\equiv v\equiv 1 \pmod 4$, by [S3, Theorem 4.3] we have

$$(-11)^{[\frac p4]}\equiv\begin{cases} (\frac{u}{11}) \pmod p & \text{if } p\equiv 1 \pmod 4,\\ (\frac{u}{11})\frac{v}{u} \pmod p & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

If $u\equiv v\equiv 0 \pmod 2$, by [S3, Corollary 4.6] we have

$$11^{[\frac p4]}\equiv\begin{cases}-(\frac{u}{11}) \pmod p&\text{if } p\equiv 1 \pmod 4 \text{ and } 8\mid u-2,\\ -(\frac{u}{11})\frac vu \pmod p&\text{if } p\equiv 3 \pmod 4 \text{ and } 8\mid v-2.\end{cases}$$

Now combining all the above we derive the result.

From [RPR], [JM] and [PV] we know that for any prime $p>3$,

$$(2.9)\quad \begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3-8\cdot 19x+2\cdot 19^2}{p}\Big)\\ &=\begin{cases}(\frac 2p)(\frac u{19})u&\text{if } (\frac p{19})=1 \text{ and so } 4p=u^2+19v^2,\\ 0&\text{if } (\frac p{19})=-1,\end{cases}\\ &\sum_{x=0}^{p-1}\Big(\frac{x^3-80\cdot 43x+42\cdot 43^2}{p}\Big)\\ &=\begin{cases}(\frac 2p)(\frac u{43})u&\text{if } (\frac p{43})=1 \text{ and so } 4p=u^2+43v^2,\\ 0&\text{if } (\frac p{43})=-1,\end{cases}\\ &\sum_{x=0}^{p-1}\Big(\frac{x^3-440\cdot 67x+434\cdot 67^2}{p}\Big)\\ &=\begin{cases}(\frac 2p)(\frac u{67})u&\text{if } (\frac p{67})=1 \text{ and so } 4p=u^2+67v^2,\\ 0&\text{if } (\frac p{67})=-1,\end{cases}\\ &\sum_{x=0}^{p-1}\Big(\frac{x^3-80\cdot 23\cdot 29\cdot 163x+14\cdot 11\cdot 19\cdot 127\cdot 163^2}{p}\Big)\\ &=\begin{cases}(\frac 2p)(\frac u{163})u&\text{if } (\frac p{163})=1 \text{ and so } 4p=u^2+163v^2,\\ 0&\text{if } (\frac p{163})=-1.\end{cases}\end{aligned}$$

So, by the method of proof of Corollary 2.7 one can determine $P_{[\frac p6]}(\frac 3{32}\sqrt{114})$, $P_{[\frac p6]}(\frac{63\sqrt{645}}{1600})$, $P_{[\frac p6]}(\frac{651}{96800}\sqrt{22110})$ and $P_{[\frac p6]}(\frac{557403}{26680^2}\sqrt{1630815}) \pmod p$.

Lemma 2.3. *Let p be a prime greater than 3, and let t be a variable. Then*

$$\begin{aligned}P_{[\frac p6]}(t)&\equiv\sum_{k=0}^{[p/6]}\binom{6k}{3k}\binom{3k}{k}\Big(\frac{1-t}{864}\Big)^k\\ &\equiv\sum_{k=0}^{p-1}\binom{6k}{3k}\binom{3k}{k}\Big(\frac{1-t}{864}\Big)^k \pmod p.\end{aligned}$$

Proof. Suppose that $k\in\{0,1,\ldots,p-1\}$ and that $r\in\{1,5\}$ is given by $p \equiv r \pmod 6$. Then clearly

$$\begin{aligned}&\binom{[\frac p6]+k}{2k}\binom{2k}{k}\\&=\frac{(\frac{p-r}{6}+k)(\frac{p-r}{6}+k-1)\cdots(\frac{p-r}{6}-k+1)}{k!^2}\\&=\frac{(p+6k-r)(p+6k-6-r)\cdots(p-(6k-6)-r)}{6^{2k}\cdot k!^2}\\&\equiv(-1)^k\frac{(6k-r)(6k-6-r)\cdots(6-r)\cdot r(r+6)\cdots(6k-6+r)}{6^{2k}\cdot k!^2}\\&=\frac{(-1)^k\cdot(6k)!}{(2\cdot4\cdots6k)(3\cdot9\cdot15\cdots(6k-3))\cdot6^{2k}\cdot k!^2}\\&=\frac{(-1)^k\cdot(6k)!}{2^{3k}(3k)!\cdot3^k\frac{(2k)!}{2\cdot4\cdot6\cdots2k}\cdot36^kk!^2}=\frac{(6k)!}{(-432)^k(3k)!(2k)!k!}\pmod p.\end{aligned}$$

Hence

$$\binom{[\frac p6]}{k}\binom{[\frac p6]+k}{k}=\binom{[\frac p6]+k}{2k}\binom{2k}{k}\equiv\frac{\binom{6k}{3k}\binom{3k}{k}}{(-432)^k}\pmod p. \tag{2.10}$$

Therefore $p \mid \binom{6k}{3k}\binom{3k}{k}$ for $\frac p6 < k < p$. Now combining (2.10) with (1.3) yields the result.

Theorem 2.2. *Let $p>3$ be a prime and $m,n\in R_p$ with $m\not\equiv 0 \pmod p$. Then*

$$\begin{aligned}P_{[\frac p6]}\Big(\frac{n}{2m^3}\Big)&\equiv\sum_{k=0}^{[p/6]}\binom{6k}{3k}\binom{3k}{k}\Big(\frac{2m^3-n}{12^3m^3}\Big)^k\\&\equiv-\Big(\frac{3m}{p}\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-3m^2x+n}{p}\Big)\pmod p.\end{aligned}$$

Proof. Replacing m by $-3m^2$ in Theorem 2.1 and then applying Lemma 2.3 we deduce the result.

Corollary 2.8. *Let $p>3$ be a prime, and let $c(n)$ be given by*

$$q\prod_{k=1}^{\infty}(1-q^k)^2(1-q^{11k})^2=\sum_{n=1}^{\infty}c(n)q^n\quad(|q|<1).$$

Then

$$c(p)\equiv P_{[\frac p6]}\Big(\frac{19}{8}\Big)\equiv(-1)^{\frac{p-1}{2}}\sum_{k=0}^{[p/6]}\frac{\binom{6k}{3k}\binom{3k}{k}}{256^k}\pmod p.$$

Proof. It is easy to see that the result holds for $p=11$. Now assume $p\ne 11$. By the well known result of Eichler (see [KKS, Theorem 12.2]), we have

$$|\{(x,y)\in\mathbb{F}_p\times\mathbb{F}_p:\ y^2+y=x^3-x^2\}|=p-c(p).$$

Since

$$\begin{aligned}
&|\{(x,y)\in\mathbb{F}_p\times\mathbb{F}_p:\ y^2+y=x^3-x^2\}| \\
&=\Big|\big\{(x,y)\in\mathbb{F}_p\times\mathbb{F}_p:\ \big(y+\frac12\big)^2=x^3-x^2+\frac14\big\}\Big| \\
&=\Big|\big\{(x,y)\in\mathbb{F}_p\times\mathbb{F}_p:\ y^2=x^3-x^2+\frac14\big\}\Big| \\
&=p+\sum_{x=0}^{p-1}\Big(\frac{x^3-x^2+\frac14}{p}\Big)=p+\sum_{x=0}^{p-1}\Big(\frac{(x+\frac13)^3-(x+\frac13)^2+\frac14}{p}\Big) \\
&=p+\sum_{x=0}^{p-1}\Big(\frac{x^3-\frac13x+\frac{19}{108}}{p}\Big)=p+\sum_{x=0}^{p-1}\Big(\frac{(\frac x6)^3-\frac13\cdot\frac x6+\frac{19}{108}}{p}\Big) \\
&=p+\Big(\frac6p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-12x+38}{p}\Big),
\end{aligned}$$

we obtain

$$c(p)=-\Big(\frac6p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-12x+38}{p}\Big). \tag{2.11}$$

Using Theorem 2.2 we see that

$$c(p)=-\Big(\frac6p\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3-12x+38}{p}\Big)\equiv P_{[\frac p6]}\Big(\frac{19}{8}\Big)\ (\mathrm{mod}\ p).$$

From (1.2) and Lemma 2.3 we have

$$\begin{aligned}
P_{[\frac p6]}\Big(\frac{19}{8}\Big)&=(-1)^{[\frac p6]}P_{[\frac p6]}\Big(-\frac{19}{8}\Big) \\
&\equiv(-1)^{\frac{p-1}{2}}\sum_{k=0}^{[p/6]}\binom{6k}{3k}\binom{3k}{k}\Big(\frac{1+19/8}{864}\Big)^k \\
&=(-1)^{\frac{p-1}{2}}\sum_{k=0}^{[p/6]}\frac{\binom{6k}{3k}\binom{3k}{k}}{256^k}\ (\mathrm{mod}\ p).
\end{aligned}$$

Thus the result follows.

Remark 2.2 Set $q=e^{2\pi iz}$ and $f(z)=q\prod_{k=1}^{\infty}(1-q^k)^2(1-q^{11k})^2$. It is known that $f(z)$ is the unique weight 2 modular form of level 11.

Theorem 2.3. *Let $p>3$ be a prime, and let t be a variable. Then*

$$P_{[\frac p6]}(t)\equiv-\Big(\frac3p\Big)\sum_{x=0}^{p-1}(x^3-3x+2t)^{\frac{p-1}{2}}\ (\mathrm{mod}\ p). \tag{2.12}$$

Proof. Taking $m=1$ and $n=2t$ in Theorem 2.2 and applying Euler's criterion we see that (2.12) is true for $t=0,1,\dots,p-1$. Since both sides of (2.12) are polynomials in t of degree at most $(p-1)/2$, applying Lagrange's theorem we conclude that (2.12) holds when t is a variable.

Theorem 2.4. *Let $p>3$ be a prime and let t be a variable. Then*

$$P_{\frac{p-1}{2}}(t) \equiv \begin{cases} (-t^2-3)^{\frac{p-1}{4}} P_{[\frac p6]}(\frac{t(t^2-9)\sqrt{t^2+3}}{(t^2+3)^2}) \pmod p & \text{if } 4\mid p-1,\\ -\frac{(-t^2-3)^{\frac{p+1}{4}}}{\sqrt{t^2+3}} P_{[\frac p6]}(\frac{t(t^2-9)\sqrt{t^2+3}}{(t^2+3)^2}) \pmod p & \text{if } 4\mid p-3 \end{cases}$$

and

$$P_{[\frac p4]}(t) \equiv \begin{cases} (6t+10)^{\frac{p-1}{4}} P_{[\frac p6]}(\frac{(9t+7)\sqrt{6t+10}}{(3t+5)^2}) \pmod p & \text{if } 4\mid p-1,\\ \frac{(6t+10)^{\frac{p+1}{4}}}{\sqrt{6t+10}} P_{[\frac p6]}(\frac{(9t+7)\sqrt{6t+10}}{(3t+5)^2}) \pmod p & \text{if } 4\mid p-3. \end{cases}$$

Proof. By (1.1), both sides of the two congruences are polynomials in t of degree at most $p-3$. By Lagrange's theorem, it suffices to show that the congruences are true for $p-2$ values of $t\in\{0,1,\ldots,p-1\}$. Now combining (1.4) and (1.6) with Theorem 2.1 we deduce the result.

Corollary 2.9. *Let $p>3$ be a prime and $m\in R_p$ with $m\not\equiv 0 \pmod p$. Then*

$$\begin{aligned} &P_{[\frac p4]}\Big(\frac{2m^2-5}{3}\Big)\\ &\equiv \Big(\frac{2m}{p}\Big)P_{[\frac p6]}\Big(\frac{3m^2-4}{m^3}\Big) \equiv \Big(\frac{-2}{p}\Big)\sum_{k=0}^{[p/4]}\binom{4k}{2k}\binom{2k}{k}\Big(\frac{m^2-1}{192}\Big)^k\\ &\equiv \Big(\frac{2m}{p}\Big)\sum_{k=0}^{[p/6]}\binom{6k}{3k}\binom{3k}{k}\Big(\frac{(m+1)(m-2)^2}{864m^3}\Big)^k \pmod p. \end{aligned}$$

Proof. Taking $t=(2m^2-5)/3$ in Theorem 2.4 and then applying [S6, Lemma 2.2] and Lemma 2.3 we deduce the result.

Theorem 2.5. *Let $p>3$ be a prime and let t be a variable. Then*

$$P_{[\frac p3]}(t) \equiv \begin{cases} (5-4t)^{\frac{p-1}{4}} P_{[\frac p6]}(\frac{2t^2-14t+11}{(5-4t)^2}\sqrt{5-4t}) \pmod p \\ \qquad\qquad \text{if } p\equiv 1 \pmod 4,\\ -\frac{(5-4t)^{\frac{p+1}{4}}}{\sqrt{5-4t}} P_{[\frac p6]}(\frac{2t^2-14t+11}{(5-4t)^2}\sqrt{5-4t}) \pmod p \\ \qquad\qquad \text{if } p\equiv 3 \pmod 4. \end{cases}$$

Proof. By (1.1), both sides of the congruence are polynomials in t of degree at most $p-2$. By Lagrange's theorem, it suffices to show that the congruence is true for all $t\in R_p$ with $t\not\equiv \frac 54 \pmod p$. Now assume $t\in R_p$ and $t\not\equiv \frac 54 \pmod p$. Set $m=3(4t-5)$ and $n=2(2t^2-14t+11)$. Then

$$\frac{3n\sqrt{-3m}}{2m^2}=\frac{(2t^2-14t+11)\sqrt{5-4t}}{(5-4t)^2}.$$

Thus, by (1.5) and Theorem 2.1 we have

$$\begin{aligned}P_{[\frac p3]}(t)&\equiv -\Big(\frac p3\Big)\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\\ &\equiv\begin{cases}\big(\frac p3\big)(9(5-4t))^{\frac{p-1}4}P_{[\frac p6]}\big(\frac{2t^2-14t+11}{(5-4t)^2}\sqrt{5-4t}\big)\pmod p\\ \qquad\qquad\qquad\qquad\qquad\qquad\text{if } 4\mid p-1,\\ \big(\frac p3\big)\frac{(9(5-4t))^{\frac{p+1}4}}{\sqrt{9(5-4t)}}P_{[\frac p6]}\big(\frac{2t^2-14t+11}{(5-4t)^2}\sqrt{5-4t}\big)\pmod p\\ \qquad\qquad\qquad\qquad\qquad\qquad\text{if } 4\mid p-3.\end{cases}\end{aligned}$$

For $p\equiv 1\pmod 4$ we have $9^{\frac{p-1}4}\big(\frac p3\big)\equiv\big(\frac 3p\big)\big(\frac p3\big)=1\pmod p$, For $p\equiv 3\pmod 4$ we have $9^{\frac{p+1}4}\cdot\frac13\big(\frac p3\big)\equiv\big(\frac 3p\big)\big(\frac p3\big)=-1\pmod p$. Thus the result follows.

Corollary 2.10. *Let $p>3$ be a prime and $m\in R_p$ with $m\not\equiv 0\pmod p$. Then*

$$\begin{aligned}&P_{[\frac p3]}\Big(\frac{5-m^2}{4}\Big)\\ &\equiv\Big(\frac{-m}{p}\Big)P_{[\frac p6]}\Big(\frac{m^4+18m^2-27}{8m^3}\Big)\equiv\sum_{k=0}^{[p/3]}\binom{2k}{k}\binom{3k}{k}\Big(\frac{m^2-1}{216}\Big)^k\\ &\equiv\Big(\frac{-m}{p}\Big)\sum_{k=0}^{[p/6]}\binom{6k}{3k}\binom{3k}{k}\Big(\frac{(m+1)(3-m)^3}{2^8\cdot 3^3m^3}\Big)^k\pmod p.\end{aligned}$$

Proof. Taking $t=\frac{5-m^2}{4}$ in Theorem 2.5 and then applying [S4, Lemma 2.3] and Lemma 2.3 we deduce the result.

Corollary 2.11. *Let $p>3$ be a prime. Then*

$$\begin{aligned}&P_{[\frac p3]}\Big(\frac{7\pm 3\sqrt3}{2}\Big)\\ &\equiv\begin{cases}2a(3\pm 2\sqrt3)^{\frac{p-1}4}\pmod p & \text{if } 4\mid p-1,\ p=a^2+b^2 \text{ and } 4\mid a-1,\\ 0\pmod p & \text{if } p\equiv 3\pmod 4.\end{cases}\end{aligned}$$

Proof. Set $t=(7\pm3\sqrt3)/2$. Then $2t^2-14t+11=0$. Thus, from Theorem 2.5 and the congruence for $P_{[\frac p6]}(0)$ in the proof of Theorem 2.1 we deduce that

$$P_{[\frac p3]}\Big(\frac{7\pm3\sqrt3}{2}\Big)$$

$$\equiv \begin{cases} (-9\mp 6\sqrt{3})^{\frac{p-1}{4}} P_{[\frac p6]}(0) \equiv (3\pm 2\sqrt{3})^{\frac{p-1}{4}} \binom{\frac{p-1}{2}}{\frac{p-1}{4}} \pmod p \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } 4\mid p-1, \\ -\frac{(-9\mp 6\sqrt{3})^{\frac{p+1}{4}}}{\sqrt{-9\mp 6\sqrt{3}}} P_{[\frac p6]}(0) = 0 \pmod p \qquad \text{if } 4\mid p-3. \end{cases}$$

It is well known that $\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv 2a \pmod p$ for $p\equiv 1 \pmod 4$ (see [BEW, p.269]). Thus the corollary is proved.

Theorem 2.6. *Let $p>3$ be a prime and $m,n\in R_p$ with $m\not\equiv 0 \pmod p$. Then*

$$\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)$$
$$\equiv \begin{cases} -(-3m)^{\frac{p-1}{4}} \sum_{k=0}^{[p/12]} \binom{[\frac{p}{12}]}{k}\binom{[\frac{5p}{12}]}{k}\big(\frac{4m^3+27n^2}{4m^3}\big)^k \pmod p \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if } 4\mid p-1, \\ -\frac{3n}{2m^2}(-3m)^{\frac{p+1}{4}} \sum_{k=0}^{[p/12]} \binom{[\frac{p}{12}]}{k}\binom{[\frac{5p}{12}]}{k}\big(\frac{4m^3+27n^2}{4m^3}\big)^k \pmod p \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if } 4\mid p-3. \end{cases}$$

Proof. Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial defined by

$$P_n^{(\alpha,\beta)}(x)=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x+1)^k(x-1)^{n-k}.$$

It is known (see [AAR, p.315]) that

(2.13)
$$P_{2n}(x)=P_n^{(0,-\frac12)}(2x^2-1) \quad\text{and}\quad P_{2n+1}(x)=xP_n^{(0,\frac12)}(2x^2-1).$$

For $k=1,2,\ldots$ let $(a)_k=a(a+1)\cdots(a+k-1)$. Then clearly $(a)_k=(-1)^k k!\binom{-a}{k}$. From [B, p.170] we know that

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \binom{n+\alpha}{n}\Big(1+\sum_{k=1}^{\infty}\frac{(-n)_k(n+\alpha+\beta+1)_k}{(\alpha+1)_k\cdot k!}\Big(\frac{1-x}{2}\Big)^k\Big) \\ &= \binom{n+\alpha}{n}\sum_{k=0}^{n}\frac{\binom nk\binom{-n-\alpha-\beta-1}{k}}{\binom{-1-\alpha}{k}}\Big(\frac{x-1}{2}\Big)^k. \end{aligned}$$

Thus,

$$P_n^{(0,\beta)}(x)=\sum_{k=0}^{n}\binom nk\binom{-n-\beta-1}{k}\Big(\frac{1-x}{2}\Big)^k. \tag{2.14}$$

Hence, if $p\equiv 1 \pmod 4$, then $[\frac p6]=2[\frac{p}{12}]$ and so

$$P_{[\frac p6]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big)$$

$$
\begin{aligned}
&= P^{(0,-\frac12)}_{[\frac{p}{12}]}\Big(2\cdot\frac{-27n^2}{4m^3}-1\Big) = \sum_{k=0}^{[\frac{p}{12}]}\binom{[\frac{p}{12}]}{k}\binom{-\frac12-[\frac{p}{12}]}{k}\Big(1+\frac{27n^2}{4m^3}\Big)^k \\
&\equiv \sum_{k=0}^{[\frac{p}{12}]}\binom{[\frac{p}{12}]}{k}\binom{\frac{p-1}{2}-[\frac{p}{12}]}{k}\Big(\frac{4m^3+27n^2}{4m^3}\Big)^k \\
&= \sum_{k=0}^{[\frac{p}{12}]}\binom{[\frac{p}{12}]}{k}\binom{[\frac{5p}{12}]}{k}\Big(\frac{4m^3+27n^2}{4m^3}\Big)^k \pmod p;
\end{aligned}
$$

if $p\equiv 3 \pmod 4$, then $[\frac p6]=2[\frac{p}{12}]+1$ and so

$$
\begin{aligned}
&P_{[\frac p6]}\Big(\frac{3n\sqrt{-3m}}{2m^2}\Big) \\
&= \frac{3n\sqrt{-3m}}{2m^2}P^{(0,\frac12)}_{[\frac{p}{12}]}\Big(2\cdot\frac{-27n^2}{4m^3}-1\Big) \\
&= \frac{3n\sqrt{-3m}}{2m^2}\sum_{k=0}^{[\frac{p}{12}]}\binom{[\frac{p}{12}]}{k}\binom{-\frac32-[\frac{p}{12}]}{k}\Big(1+\frac{27n^2}{4m^3}\Big)^k \\
&\equiv \frac{3n\sqrt{-3m}}{2m^2}\sum_{k=0}^{[\frac{p}{12}]}\binom{[\frac{p}{12}]}{k}\binom{\frac{p-3}{2}-[\frac{p}{12}]}{k}\Big(\frac{4m^3+27n^2}{4m^3}\Big)^k \\
&= \frac{3n\sqrt{-3m}}{2m^2}\sum_{k=0}^{[\frac{p}{12}]}\binom{[\frac{p}{12}]}{k}\binom{[\frac{5p}{12}]}{k}\Big(\frac{4m^3+27n^2}{4m^3}\Big)^k \pmod p.
\end{aligned}
$$

Now combining the above with Theorem 2.1 we deduce the result.

For a prime p and $a\in R_p$ let $\langle a\rangle_p$ denote the unique integer $a_0\in\{0,1,\ldots,p-1\}$ such that $a\equiv a_0 \pmod p$.

Lemma 2.4. *Let $p>3$ be a prime and let $t\in R_p$ with $t\not\equiv 0 \pmod p$. Then*

$$
\begin{aligned}
&\sum_{k=0}^{p-1}\binom{-\frac{1}{12}}{k}\binom{-\frac{5}{12}}{k}(1-t)^k \\
&\equiv t^{\langle -\frac{1}{12}\rangle_p}\sum_{k=0}^{p-1}\binom{-\frac{1}{12}}{k}\binom{-\frac{7}{12}}{k}\Big(1-\frac1t\Big)^k \\
&\equiv \begin{cases} P_{[\frac p6]}(\sqrt t) \pmod p & \text{if } p\equiv 1 \pmod 4,\\ (\frac tp)P_{[\frac p6]}(\sqrt t)\sqrt t \pmod p & \text{if } p\equiv 3 \pmod 4.\end{cases}
\end{aligned}
$$

Proof. Taking $a=-\frac{1}{12}$ in [S7, Theorem 2.2] and then applying

[S7, Lemmas 2.2-2.3] we see that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{-\frac{1}{12}}{k}\binom{-\frac{5}{12}}{k}(1-t)^k\\&\equiv t^{\langle-\frac{1}{12}\rangle_p}\sum_{k=0}^{p-1}\binom{-\frac{1}{12}}{k}\binom{-\frac{7}{12}}{k}\Big(\frac{t-1}{t}\Big)^k\equiv P_{2\langle-\frac{1}{12}\rangle_p}(\sqrt{t})\\&=\begin{cases}P_{\frac{p-1}{6}}(\sqrt{t})\pmod p&\text{if } 12\mid p-1,\\P_{\frac{5p-1}{6}}(\sqrt{t})\equiv P_{\frac{p-5}{6}}(\sqrt{t})\pmod p&\text{if } 12\mid p-5,\\P_{\frac{7p-1}{6}}(\sqrt{t})\equiv(\sqrt{t})^pP_{\frac{p-1}{6}}(\sqrt{t})\pmod p&\text{if } 12\mid p-7,\\P_{\frac{11p-1}{6}}(\sqrt{t})\equiv(\sqrt{t})^pP_{\frac{5p-1}{6}}(\sqrt{t})\equiv(\sqrt{t})^pP_{\frac{p-5}{6}}(\sqrt{t})&\pmod p\\&\text{if } 12\mid p-11.\end{cases}\end{aligned}$$

To see the result, we note that $(\sqrt{t})^p=t^{\frac{p-1}{2}}\sqrt{t}\equiv(\frac{t}{p})\sqrt{t}\pmod p$.

Theorem 2.7. *Let $p>3$ be a prime and $m,n\in R_p$ with $mn\not\equiv 0\pmod p$. Then*

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\\&\equiv\begin{cases}-(-3m)^{\frac{p-1}{4}}\sum_{k=0}^{p-1}\binom{-\frac{1}{12}}{k}\binom{-\frac{5}{12}}{k}(\frac{4m^3+27n^2}{4m^3})^k\pmod p\\ \qquad\qquad \text{if } 4\mid p-1,\\ \frac{2m}{9n}(\frac{-3m}{p})(-3m)^{\frac{p+1}{4}}\sum_{k=0}^{p-1}\binom{-\frac{1}{12}}{k}\binom{-\frac{5}{12}}{k}(\frac{4m^3+27n^2}{4m^3})^k\pmod p\\ \qquad\qquad \text{if } 4\mid p-3.\end{cases}\\&\equiv\begin{cases}(-1)^{\frac{p+1}{2}}(\frac{n}{2})^{\frac{p-1}{6}}\sum_{k=0}^{p-1}\binom{-\frac{1}{12}}{k}\binom{-\frac{7}{12}}{k}(\frac{4m^3+27n^2}{27n^2})^k\pmod p\\ \qquad\qquad \text{if } 3\mid p-1,\\ (-1)^{\frac{p+1}{2}}\frac{3}{m}(\frac{2}{n})^{\frac{p-5}{6}}\sum_{k=0}^{p-1}\binom{-\frac{1}{12}}{k}\binom{-\frac{7}{12}}{k}(\frac{4m^3+27n^2}{27n^2})^k\pmod p\\ \qquad\qquad \text{if } 3\mid p-2.\end{cases}\end{aligned}$$

Proof. Set $t=-\frac{27n^2}{4m^3}$. Then $1-t=\frac{4m^3+27n^2}{4m^3}$. Since

$$\begin{aligned}&t^{\langle-\frac{1}{12}\rangle_p}\\&=\begin{cases}(-\frac{27n^2}{4m^3})^{\frac{p-1}{12}}=(-\frac{3}{m})^{\frac{p-1}{4}}(\frac{n}{2})^{\frac{p-1}{6}}&\text{if } 12\mid p-1,\\(-\frac{27n^2}{4m^3})^{\frac{5p-1}{12}}\equiv(\frac{-3m}{p})(-\frac{m}{3})^{\frac{p-5}{4}}(\frac{2}{n})^{\frac{p-5}{6}}\pmod p&\text{if } 12\mid p-5,\\(-\frac{27n^2}{4m^3})^{\frac{7p-1}{12}}\equiv(\frac{-3m}{p})(-\frac{3}{m})^{\frac{p+5}{4}}(\frac{n}{2})^{\frac{p+5}{6}}\pmod p&\text{if } 12\mid p-7,\\(-\frac{27n^2}{4m^3})^{\frac{11p-1}{12}}\equiv(-\frac{m}{3})^{\frac{p-11}{4}}(\frac{2}{n})^{\frac{p-11}{6}}\pmod p&\text{if } 12\mid p-11,\end{cases}\end{aligned}$$

using Lemma 2.4 and Theorem 2.1 we deduce the result.

3. A general congruence modulo p^2

Lemma 3.1. *For any nonnegative integer n we have*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\binom{k}{n-k}(-432)^{n-k}$$
$$=\sum_{k=0}^{n}\binom{3k}{k}\binom{6k}{3k}\binom{3(n-k)}{n-k}\binom{6(n-k)}{3(n-k)}.$$

Proof. Let m be a nonnegative integer. For $k\in\{0,1,\ldots,m\}$ set

$$F_1(m,k)=\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\binom{k}{m-k}(-432)^{m-k},$$
$$F_2(m,k)=\binom{3k}{k}\binom{6k}{3k}\binom{3(m-k)}{m-k}\binom{6(m-k)}{3(m-k)}.$$

For $k\in\{0,1,\ldots,m+1\}$ set

$$G_1(m,k)=-k^2(m+2)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\binom{k}{m+2-k}(-432)^{m+2-k},$$
$$G_2(m,k)=\frac{12k^2(36m^2-36km+129m-62k+114)}{(m+2-k)^2}$$
$$\times\binom{3k}{k}\binom{6k}{3k}\binom{3(m+1-k)}{m+1-k}\binom{6(m+1-k)}{3(m+1-k)}.$$

For $i=1,2$ and $k\in\{0,1,\ldots,m\}$, using Maple it is easy to check that

(3.1)
$$\begin{aligned}&(m+2)^3F_i(m+2,k)-24(2m+3)(18m^2+54m+41)F_i(m+1,k)\\&\qquad+20736(m+1)(3m+1)(3m+5)F_i(m,k)\\&=G_i(m,k+1)-G_i(m,k).\end{aligned}$$

Set $S_i(n)=\sum_{k=0}^{n}F_i(n,k)$ for $n=0,1,2,\ldots$. Then

$$\begin{aligned}&(m+2)^3(S_i(m+2)-F_i(m+2,m+2)-F_i(m+2,m+1))\\&\quad-24(2m+3)(18m^2+54m+41)(S_i(m+1)-F_i(m+1,m+1))\\&\quad+20736(m+1)(3m+1)(3m+5)S_i(m)\\&=\sum_{k=0}^{m}\Big((m+2)^3F_i(m+2,k)\\&\qquad-24(2m+3)(18m^2+54m+41)F_i(m+1,k)\\&\qquad+20736(m+1)(3m+1)(3m+5)F_i(m,k)\Big)\\&=\sum_{k=0}^{m}(G_i(m,k+1)-G_i(m,k))=G_i(m,m+1)-G_i(m,0)\end{aligned}$$

$$= G_i(m, m+1).$$

Thus, for $i = 1, 2$ and $m = 0, 1, 2, \ldots$,

(3.2)

$$\begin{aligned}(m+2)^3 S_i(m+2) &- 24(2m+3)(18m^2+54m+41)S_i(m+1)\\ &+ 20736(m+1)(3m+1)(3m+5)S_i(m)\\ = G_i(m, m+1) &+ (m+2)^3(F_i(m+2, m+2) + F_i(m+2, m+1))\\ &- 24(2m+3)(18m^2+54m+41)F_i(m+1, m+1) = 0.\end{aligned}$$

Since $S_1(0) = 1 = S_2(0)$ and $S_1(1) = 120 = S_2(1)$, from (3.2) we deduce $S_1(n) = S_2(n)$ for all $n = 0, 1, 2, \ldots$. This completes the proof.

For any prime p and integer n, if $p^\alpha \mid n$ but $p^{\alpha+1} \nmid n$, we write $p^\alpha \parallel n$.

Lemma 3.2. *Let p be an odd prime and $k, r \in \{0, 1, \ldots, p-1\}$ with $k + r \ge p$. Then*

$$\binom{3k}{k}\binom{6k}{3k}\binom{3r}{r}\binom{6r}{3r} \equiv 0 \pmod{p^2}.$$

Proof. For any positive integer n we have $\binom{3n}{n} = 3\binom{3n-1}{n-1}$. Thus the result is true for $p = 3$. Now assume $p > 3$. By (2.10), $p \mid \binom{3n}{n}\binom{6n}{3n}$ for $\frac{p}{6} < n \le p-1$. Thus, if $k > \frac{p}{6}$ and $r > \frac{p}{6}$, then $p^2 \mid \binom{3k}{k}\binom{6k}{3k}\binom{3r}{r}\binom{6r}{3r}$. If $r < \frac{p}{6}$, then $k \ge p - r > \frac{5p}{6}$, $p^5 \mid (6k)!$, $p \parallel (2k)!$, $p^2 \parallel (3k)!$ and so $\binom{3k}{k}\binom{6k}{3k} = \frac{(6k)!}{k!(2k)!(3k)!} \equiv 0 \pmod{p^2}$. Similarly, if $k < \frac{p}{6}$, then $r > \frac{5p}{6}$ and so $\binom{3r}{r}\binom{6r}{3r} \equiv 0 \pmod{p^2}$. Thus the lemma is proved.

Theorem 3.1. *Let p be an odd prime and let x be a variable. Then*

$$\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}(x(1-432x))^k \equiv \Big(\sum_{k=0}^{p-1}\binom{3k}{k}\binom{6k}{3k}x^k\Big)^2 \pmod{p^2}.$$

Proof. For $\frac{p}{2} < k < p$ we have $p \mid \binom{2k}{k}$ and $p \mid \binom{3k}{k}\binom{6k}{3k}$ by (2.10). Thus $p^2 \mid \binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}$ for $\frac{p}{2} < k < p$. Hence, using Lemma 3.1 we deduce that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}(x(1-432x))^k\\ &\equiv \sum_{k=0}^{(p-1)/2}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}x^k(1-432x)^k\\ &= \sum_{k=0}^{(p-1)/2}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}x^k\sum_{r=0}^{k}\binom{k}{r}(-432x)^r\\ &= \sum_{m=0}^{p-1}x^m\sum_{k=0}^{m}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\binom{k}{m-k}(-432)^{m-k}\end{aligned}$$

$$= \sum_{m=0}^{p-1} x^m \sum_{k=0}^{m} \binom{3k}{k}\binom{6k}{3k}\binom{3(m-k)}{m-k}\binom{6(m-k)}{3(m-k)}$$
$$= \sum_{k=0}^{p-1} \binom{3k}{k}\binom{6k}{3k} x^k \sum_{m=k}^{p-1} \binom{3(m-k)}{m-k}\binom{6(m-k)}{3(m-k)} x^{m-k}$$
$$= \sum_{k=0}^{p-1} \binom{3k}{k}\binom{6k}{3k} x^k \sum_{r=0}^{p-1-k} \binom{3r}{r}\binom{6r}{3r} x^r$$
$$= \Big(\sum_{k=0}^{p-1} \binom{3k}{k}\binom{6k}{3k} x^k\Big)^2$$
$$- \sum_{k=0}^{p-1} \binom{3k}{k}\binom{6k}{3k} x^k \sum_{r=p-k}^{p-1} \binom{3r}{r}\binom{6r}{3r} x^r \pmod{p^2}.$$

By Lemma 3.2, $p^2 \mid \binom{3k}{k}\binom{6k}{3k}\binom{3r}{r}\binom{6r}{3r}$ for $0 \le k \le p-1$ and $p-k \le r \le p-1$. Thus

$$\sum_{k=0}^{p-1} \binom{3k}{k}\binom{6k}{3k} x^k \sum_{r=p-k}^{p-1} \binom{3r}{r}\binom{6r}{3r} x^r \equiv 0 \pmod{p^2}.$$

Now combining all the above we obtain the result.

Corollary 3.1. *Let p be a prime greater than 3 and $m \in R_p$ with $m \not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k} \equiv \Big(\sum_{k=0}^{p-1} \binom{3k}{k}\binom{6k}{3k}\Big(\frac{1-\sqrt{1-\frac{1728}{m}}}{864}\Big)^k\Big)^2 \pmod{p^2}.$$

Proof. Taking $x = \frac{1-\sqrt{1-1728/m}}{864}$ in Theorem 3.1 we deduce the result.

Corollary 3.2. *Let p be a prime greater than 3 and let t be a variable. Then*

$$t\sum_{k=0}^{p-1} \binom{2k}{k}\binom{3k}{k}\binom{6k}{3k} k\Big(\frac{1-t^2}{1728}\Big)^k$$
$$\equiv (t+1)\Big(\sum_{k=0}^{p-1} \binom{3k}{k}\binom{6k}{3k}\Big(\frac{1-t}{864}\Big)^k\Big)$$
$$\times \Big(\sum_{k=0}^{p-1} \binom{3k}{k}\binom{6k}{3k} k\Big(\frac{1-t}{864}\Big)^k\Big) \pmod{p^2}.$$

Proof. Putting $x = \frac{1-t}{864}$ in Theorem 3.1 we see that

$$\sum_{k=0}^{p-1} \binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{1-t^2}{1728}\Big)^k$$
$$= \Big(\sum_{k=0}^{p-1} \binom{3k}{k}\binom{6k}{3k}\Big(\frac{1-t}{864}\Big)^k\Big)^2 + p^2 f(t),$$

where $f(t)$ is a polynomial in t with coefficients in R_p. Taking derivatives and then multiplying by $\frac{1-t^2}{1728}$ on both sides we deduce the result.

Lemma 3.3. *Let p be a prime of the form $4k+1$ and $p = a^2+b^2 (a,b\in\mathbb{Z})$ with $a\equiv 1 \pmod 4$. Then*

$$P_{[\frac p6]}(0)\equiv \binom{\frac{p-1}{2}}{[\frac{p}{12}]}\equiv\begin{cases}2a \pmod p & \text{if } 12\mid p-1 \text{ and } 3\nmid a,\\ -2a \pmod p & \text{if } 12\mid p-1 \text{ and } 3\mid a,\\ 2b \pmod p & \text{if } 12\mid p-5 \text{ and } 3\mid a-b.\end{cases}$$

Proof. By Lemma 2.1(i) and the proof of Theorem 2.1,

$$P_{[\frac p6]}(0)=\frac{1}{(-4)^{[\frac{p}{12}]}}\binom{[\frac p6]}{[\frac{p}{12}]}\equiv\binom{\frac{p-1}{2}}{[\frac{p}{12}]}\equiv(-3)^{-\frac{p-1}{4}}\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \pmod p.$$

By Gauss' congruence ([BEW, p.269]), $\binom{\frac{p-1}{2}}{\frac{p-1}{4}}\equiv 2a \pmod p$. By [S1, Theorem 2.2 and Example 2.1],

$$(-3)^{\frac{p-1}{4}}\equiv\begin{cases}1 \pmod p & \text{if } 12\mid p-1 \text{ and } 3\nmid a,\\ -1 \pmod p & \text{if } 12\mid p-1 \text{ and } 3\mid a,\\ -\frac ba\equiv\frac ab \pmod p & \text{if } 12\mid p-5 \text{ and } 3\mid a-b.\end{cases}$$

Thus the result follows.

We note that for primes $p\equiv 1 \pmod{12}$, the congruence $\binom{(p-1)/2}{(p-1)/12}\equiv\pm 2a \pmod p$ was given in [HW, Corollary 4.2.2].

Let $p>3$ be a prime. By the work of Mortenson[M] and Zhi-Wei Sun[Su2],

(3.3)
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k}\equiv\begin{cases}(\frac p3)(4a^2-2p) \pmod{p^2} & \text{if } p=a^2+b^2\equiv 1 \pmod 4 \text{ and } 2\nmid a,\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

In [Su1, Conjecture B16] Zhi-Wei Sun conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{864^k}\equiv\begin{cases}0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\\ (-1)^{[\frac a6]}(2a-\frac{p}{2a}) \pmod{p^2} & \text{if } 12\mid p-1,\ p=a^2+b^2 \text{ and } 4\mid a-1,\\ (\frac{ab}{3})(2b-\frac{p}{2b}) \pmod{p^2} & \text{if } 12\mid p-5,\ p=a^2+b^2 \text{ and } 4\mid a-1.\end{cases}$$

In [Su4], Zhi-Wei Sun confirmed the conjecture in the case $p \equiv 3 \pmod 4$.

Now we prove the above conjecture for primes $p \equiv 1 \pmod 4$.

Theorem 3.2. *Let p be a prime of the form $4k+1$ and so $p = a^2 + b^2$ with $a, b \in \mathbb{Z}$ and $a \equiv 1 \pmod 4$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{864^k} \equiv \begin{cases} 2a - \frac{p}{2a} \pmod{p^2} & \text{if } 12 \mid p-1 \text{ and } 3 \nmid a, \\ -2a + \frac{p}{2a} \pmod{p^2} & \text{if } 12 \mid p-1 \text{ and } 3 \mid a, \\ 2b - \frac{p}{2b} \pmod{p^2} & \text{if } 12 \mid p-5 \text{ and } 3 \mid a-b. \end{cases}$$

Proof. From Lemma 3.3 we have $P_{[\frac{p}{6}]}(0) \equiv 2r \pmod p$, where

$$r = \begin{cases} a & \text{if } p \equiv 1 \pmod{12} \text{ and } 3 \nmid a, \\ -a & \text{if } p \equiv 1 \pmod{12} \text{ and } 3 \mid a, \\ b & \text{if } p \equiv 5 \pmod{12} \text{ and } 3 \mid a-b. \end{cases}$$

By (2.10), $p \mid \binom{6k}{3k}\binom{3k}{k}$ for $\frac{p}{6} < k < p$. Thus, applying Lemma 2.3 and the above we get

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{864^k} \equiv P_{[\frac{p}{6}]}(0) \equiv 2r \pmod p.$$

Set $\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{864^k} = 2r + qp$. Using Corollary 3.1 we see that

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k} &\equiv \Big(\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{864^k}\Big)^2 \\ &= (2r+qp)^2 \equiv 4r^2 + 4rqp \pmod{p^2}. \end{aligned}$$

Thus, applying (3.3) we obtain $(\frac{p}{3})(4a^2 - 2p) \equiv 4r^2 + 4rqp \pmod{p^2}$. Hence $q \equiv -\frac{1}{2r} \pmod p$ and the proof is complete.

4. Congruences for $\sum_{k=0}^{\frac{p-1}{2}} \binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}/m^k$

Theorem 4.1. *Let $p > 3$ be a prime, $m \in R_p$, $m \not\equiv 0 \pmod p$ and $t = \sqrt{1 - 1728/m}$. Then*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k} \equiv P_{[\frac{p}{6}]}(t)^2 \equiv \Big(\sum_{x=0}^{p-1} (x^3 - 3x + 2t)^{\frac{p-1}{2}}\Big)^2 \pmod p.$$

Moreover, if $P_{[\frac{p}{6}]}(t) \equiv 0 \pmod p$ or $\sum_{x=0}^{p-1}(x^3 - 3x + 2t)^{\frac{p-1}{2}} \equiv 0 \pmod p$, then

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k} \equiv 0 \pmod{p^2}$$

and

$$\sum_{k=0}^{[p/6]}\frac{k\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}\equiv 0 \pmod p \quad \textit{for} \quad m\not\equiv 1728 \pmod p.$$

Proof. For $k\in\{\frac{p+1}{2},\dots,p-1\}$ we have $p\mid\binom{2k}{k}$ and $p\mid\binom{3k}{k}\binom{6k}{3k}$ by (2.10), thus $p^2\mid\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}$. Since $\frac{1-t}{864}(1-432\cdot\frac{1-t}{864})=\frac{1-t^2}{1728}=\frac{1}{m}$, by Theorem 3.1 we have

(4.1)

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}\\ &\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}\equiv\Big(\sum_{k=0}^{p-1}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{1-t}{864}\Big)^k\Big)^2 \pmod{p^2}.\end{aligned}$$

Using Lemma 2.3 and Theorem 2.3 we see that

$$\begin{aligned}&\sum_{k=0}^{p-1}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{1-t}{864}\Big)^k\\ &\equiv P_{[\frac{p}{6}]}(t)\equiv-\Big(\frac{3}{p}\Big)\sum_{x=0}^{p-1}(x^3-3x+2t)^{\frac{p-1}{2}} \pmod p.\end{aligned}$$

This together with (4.1), Corollary 3.2 and the fact that $p\mid\binom{3k}{k}\binom{6k}{3k}$ for $\frac{p}{6}<k<p$ yields the result.

Theorem 4.2. *Let $p>3$ be a prime and $m,n\in R_p$ with $m\not\equiv 0 \pmod p$. Then*

$$\begin{aligned}&\Big(\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\Big)^2\\ &\equiv\Big(\frac{-3m}{p}\Big)\sum_{k=0}^{(p-1)/2}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{4m^3+27n^2}{12^3\cdot 4m^3}\Big)^k\\ &\equiv\Big(\frac{-3m}{p}\Big)\sum_{k=0}^{[p/6]}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{4m^3+27n^2}{12^3\cdot 4m^3}\Big)^k \pmod p.\end{aligned}$$

Moreover, if $\sum_{x=0}^{p-1}(\frac{x^3+mx+n}{p})=0$, then

$$\sum_{k=0}^{(p-1)/2}\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{4m^3+27n^2}{12^3\cdot 4m^3}\Big)^k\equiv 0 \pmod{p^2}$$

and

$$\sum_{k=0}^{[p/6]}k\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}\Big(\frac{4m^3+27n^2}{12^3\cdot 4m^3}\Big)^k\equiv 0 \pmod p \quad \textit{for} \quad n\not\equiv 0 \pmod p.$$

Proof. We first assume that $4m^3+27n^2\equiv 0\pmod p$. Clearly $-3m\equiv (\frac{9n}{2m})^2\pmod p$ and so $(\frac{-3m}{p})=1$. As $x^3+mx+n\equiv (x-\frac{3n}{m})(x+\frac{3n}{2m})^2\pmod p$ we see that

$$\begin{aligned}&\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\\&=\sum_{x=0}^{p-1}\Big(\frac{(x-\frac{3n}{m})(x+\frac{3n}{2m})^2}{p}\Big)=\sum_{\substack{x=0\\ x\not\equiv -\frac{3n}{2m}\pmod p}}^{p-1}\Big(\frac{x-\frac{3n}{m}}{p}\Big)\\&=\sum_{t=0}^{p-1}\Big(\frac{t}{p}\Big)-\Big(\frac{-\frac{3n}{2m}-\frac{3n}{m}}{p}\Big)=-\Big(\frac{-2mn}{p}\Big).\end{aligned}$$

Since $m\not\equiv 0\pmod p$ we have $n\not\equiv 0\pmod p$ and so $\sum_{x=0}^{p-1}(\frac{x^3+mx+n}{p})=-(\frac{-2mn}{p})=\pm 1$. Thus the result holds in this case.

Now we assume that $4m^3+27n^2\not\equiv 0\pmod p$. Set $t=\frac{3n\sqrt{-3m}}{2m^2}$ and $m_1=\frac{1728\cdot 4m^3}{4m^3+27n^2}$. Then $t=\sqrt{1-1728/m_1}$. From Theorems 2.1 and 4.1 we have

$$\begin{aligned}&\Big(\sum_{x=0}^{p-1}\Big(\frac{x^3+mx+n}{p}\Big)\Big)^2\\&\equiv(-3m)^{\frac{p-1}{2}}P_{[\frac p6]}(t)^2\equiv\Big(\frac{-3m}{p}\Big)\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m_1^k}\pmod p.\end{aligned}$$

By (2.10), $p\mid\binom{3k}{k}\binom{6k}{3k}$ for $\frac p6<k<p$. If $\sum_{x=0}^{p-1}(\frac{x^3+mx+n}{p})=0$, we must have $P_{[\frac p6]}(t)\equiv 0\pmod p$. Thus, applying Theorem 4.1 we see that

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m_1^k}\equiv 0\pmod{p^2}$$

and

$$\sum_{k=0}^{[p/6]}\frac{k\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m_1^k}\equiv 0\pmod p\quad\text{for}\quad n\not\equiv 0\pmod p.$$

This completes the proof.

Theorem 4.3. (*See [S4, Conjecture 2.4].*) *Let p be a prime such that $p\ne 2,3,11$. Then*

$$\begin{aligned}&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{66^{3k}}\\&\equiv\begin{cases}0\pmod{p^2}&\text{if } p\equiv 3\pmod 4,\\ (\frac{p}{33})4a^2\pmod p&\text{if } p=a^2+b^2\equiv 1\pmod 4\text{ and } 2\nmid a\end{cases}\end{aligned}$$

and

$$\sum_{k=0}^{[p/6]}\frac{k\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{66^{3k}}\equiv 0 \pmod p \quad for \quad p\equiv 3 \pmod 4 \quad with \quad p\neq 7.$$

Proof. Taking $m=-11$ and $n=14$ in Theorem 4.2 and then applying (2.2) we deduce the result.

Theorem 4.4. (*See [S4, Conjecture 2.5].) Let* $p>5$ *be a prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{20^{3k}}$$
$$\equiv\begin{cases}0 \pmod{p^2} & if\ p\equiv 5,7 \pmod 8,\\ (\frac{-5}{p})4c^2 \pmod p & if\ p=c^2+2d^2\equiv 1,3 \pmod 8\end{cases}$$

and

$$\sum_{k=0}^{[p/6]}\frac{k\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{20^{3k}}\equiv 0 \pmod p\ for\ p\equiv 5,7 \pmod 8\ with\ p\neq 7.$$

Proof. Taking $m=-30$ and $n=56$ in Theorem 4.2 and then applying the formula for $\sum_{x=0}^{p-1}(\frac{x^3-30x+56}{p})$ in the proof of Corollary 2.2 we deduce the result.

Theorem 4.5. (*See [S4, Conjecture 2.6].) Let* $p>5$ *be a prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{54000^{k}}$$
$$\equiv\begin{cases}0 \pmod{p^2} & if\ p\equiv 2 \pmod 3,\\ (\frac{p}{5})4A^2 \pmod p & if\ p=A^2+3B^2\equiv 1 \pmod 3\end{cases}$$

and

$$\sum_{k=0}^{[p/6]}\frac{k\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{54000^{k}}\equiv 0 \pmod p \quad for \quad p\equiv 2 \pmod 3 \quad with \quad p\neq 11.$$

Proof. Taking $m=-15$ and $n=22$ in Theorem 4.2 and then applying (2.3) we deduce the result.

Theorem 4.6. (*See [S4, Conjecture 2.7].) Let* $p>5$ *be a prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-12288000)^{k}}$$
$$\equiv\begin{cases}0 \pmod{p^2} & if\ 3\mid p-2,\\ (\frac{10}{p})L^2 \pmod p & if\ 3\mid p-1\ and\ so\ 4p=L^2+27M^2\end{cases}$$

and

$$\sum_{k=0}^{[p/6]}\frac{k\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-12288000)^k}\equiv 0 \pmod p\ \textit{for}\ p\equiv 2 \pmod 3\ \textit{with}\ p\ne 11,23.$$

Proof. Taking $m=-120$ and $n=506$ in Theorem 4.2 and then applying (2.4) we deduce the result.

Theorem 4.7. (*See [S4, Conjectures 2.8-2.9].*) *Let $p>7$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-15)^{3k}} \equiv \begin{cases} 0 \pmod{p^2} & \textit{if}\ p\equiv 3,5,6 \pmod 7,\\ (\frac{p}{15})4C^2 \pmod p & \textit{if}\ p=C^2+7D^2\equiv 1,2,4 \pmod 7)\end{cases}$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{255^{3k}} \equiv \begin{cases} 0 \pmod{p^2} & \textit{if}\ p\equiv 3,5,6 \pmod 7\ \textit{and}\ p\ne 17,\\ (\frac{p}{255})4C^2 \pmod p & \textit{if}\ p=C^2+7D^2\equiv 1,2,4 \pmod 7).\end{cases}$$

Proof. From (2.5), (2.7) and Theorem 4.2 we deduce the result.

Theorem 4.8. (*See [Su1, Conjecture A26].*) *Let $p\ne 2,11$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-32)^{3k}} \equiv \begin{cases} 0 \pmod{p^2} & \textit{if}\ (\frac{p}{11})=-1,\\ (\frac{-2}{p})x^2 \pmod p & \textit{if}\ (\frac{p}{11})=1\ \textit{and so}\ 4p=x^2+11y^2.\end{cases}$$

Proof. Taking $m=-96\cdot 11$ and $n=112\cdot 11^2$ in Theorem 4.2 and then applying (2.8) we deduce the result.

Similarly, from (2.9) and Theorem 4.2 we have the following result.

Theorem 4.9. (*See [Su3, Conjectures 2.8-2.10].*) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-96)^{3k}} \equiv \begin{cases} 0 \pmod{p^2} & \textit{if}\ (\frac{p}{19})=-1,\\ (\frac{-6}{p})x^2 \pmod p & \textit{if}\ (\frac{p}{19})=1\ \textit{and so}\ 4p=x^2+19y^2,\end{cases}$$

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-960)^{3k}}$$
$$\equiv\begin{cases}0\pmod{p^2} & \text{if } (\frac{p}{43})=-1 \text{ and } p\ne 5,\\ (\frac{p}{15})x^2\pmod p & \text{if } (\frac{p}{43})=1 \text{ and so } 4p=x^2+43y^2,\end{cases}$$
$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-5280)^{3k}}$$
$$\equiv\begin{cases}0\pmod{p^2} & \text{if } (\frac{p}{67})=-1 \text{ and } p\ne 5,11,\\ (\frac{-330}{p})x^2\pmod p & \text{if } (\frac{p}{67})=1 \text{ and so } 4p=x^2+67y^2\end{cases}$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-640320)^{3k}}$$
$$\equiv\begin{cases}0\pmod{p^2} & \text{if } (\frac{p}{163})=-1 \text{ and } p\ne 5,23,29,\\ (\frac{-10005}{p})x^2\pmod p & \text{if } (\frac{p}{163})=1 \text{ and so } 4p=x^2+163y^2.\end{cases}$$

Remark 4.1 From [O] we know that the only j-invariants of elliptic curves over rational field $\mathbb{Q}$ with complex multiplication are given by $0,\ 12^3,\ -15^3,\ 20^3,\ -32^3, 2\cdot 30^3, 66^3, -96^3, -3\cdot 160^3, 255^3, -960^3$, -5280^3, -640320^3, coinciding with the values of m in (3.3) and Theorems 4.3-4.9.

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